A simple, event-based, graphical derivation of the relativistic Doppler shift and aberration angle

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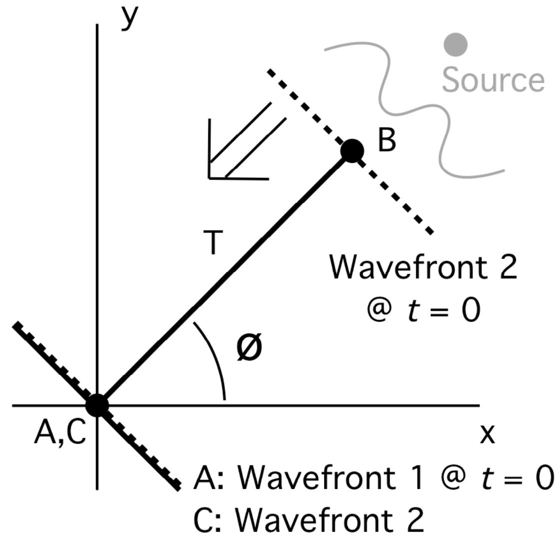
In the lab frame, two plane wavefronts move past the region near the origin having originated at a distant source that is at rest in the upper half plane as shown in the figure at right. Three events of interest are shown in the figure. Event *A* represents Wavefront 1 passing the origin at $t = 0$. Event B is a point on Wavefront 2 at $t = 0$ that *will* pass the origin one period later. Event C is that same point on Wavefront 2 passing the origin. Taking $c = 1$, the spatial separation of events A and B is both the period and wavelength in the lab frame.



Evidently, the lab (*x*,*y*,*t*) coordinates of the events are given by

$$\mathrm{A}:(0,0,0)$$
$$\mathrm{B}:(T\cos\phi, T\sin\phi, 0)$$
$$\mathrm{C}:(0,0,T)$$

Another frame moves in the –*x* direction* and has primed coordinates related to the unprimed lab frame coordinates by the Lorentz transformation:

$$x' = \gamma(x + \beta t)$$
$$y' = y$$
$$t' = \gamma(t + \beta x)$$

where β is the relative speed of the two frames and $\gamma = (1-\beta^2)^{-1/2}$. Thus, the moving frame (*x*',*y*',*t*') coordinates of the events are given by

$$\mathrm{A}':(0,0,0)$$
$$\mathrm{B}':(\gamma T\cos\phi, T\sin\phi, \gamma\beta T\cos\phi)$$
$$\mathrm{C}':(\gamma\beta T, 0, \gamma T)$$

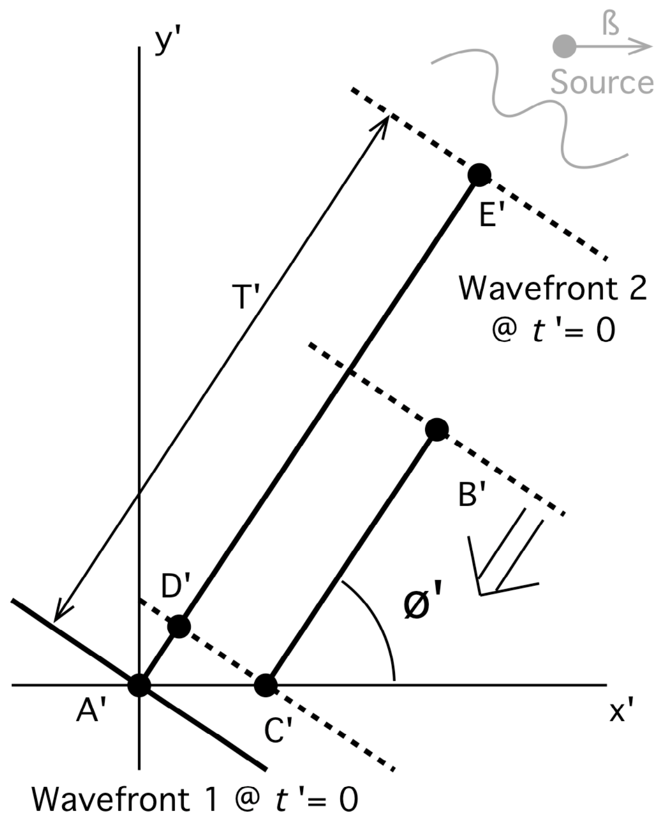
Because events B' and C' represent the same point on Wavefront 2, we can use them to identify a ray and, thus, find the direction of propagation in the moving frame as

$$\cos\phi' = \frac{x'(\mathrm{B}) - x'(\mathrm{C})}{t'(\mathrm{C}) - t'(\mathrm{B})} = \frac{\gamma T\cos\phi - \gamma\beta T}{\gamma T - \gamma\beta T\cos\phi}$$

or

$$\boxed{\cos\phi' = \frac{\cos\phi - \beta}{1 - \beta\cos\phi}}$$

This is the standard relativistic aberration formula.



To find the period in the moving frame we will first find the coordinates of event D' that takes place at the same time as C' and that is also coincident with Wavefront 2, but that sits on the ray through A'. Then we will find event E' that also sits on the ray through A' and that takes place at the same time as A', but that is coincident with Wavefront 2. The spatial separation of events A' and E' is both the period and wavelength of the wave in the moving frame.

Clearly the spatial separation of events A' and D' is related to that of events A' and C' by $\text{A'D'} = \text{A'C'}\cos\phi' = \gamma\beta T\cos\phi'$. Also, since event E' takes place a time γT earlier than event D' and lies on the same wavefront moving at the speed of light, the spatial separation of events D' and E' is $\text{D'E'} = \gamma T$. Thus,

$$T' = \text{A'D'} + \text{D'E'} = \gamma T\left(1+\beta\cos\phi'\right) = \frac{T}{\gamma\left(1-\beta\cos\phi\right)},$$

or, in terms of frequencies,

$$\nu' = \nu\gamma\left(1-\beta\cos\phi\right).$$

This is the standard relativistic Doppler shift formula.

* NOTE: The nonstandard use of a moving frame that is moving in the $-x$ direction rather than the $+x$ direction is in order to maintain consistency with the standard definition of the angles ϕ or ϕ' as the angle between the direction to the source of the waves and the direction of motion of the source. As a result the Lorentz transformation to moving coordinates takes on the more common form of the inverse transformation.